Distributivity-like Results in the Medieval Traditions of Euclid's *Elements*: Between Geometry and Arithmetic

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Contents

1. *Abstract*

The present article explores the treatment of distributivity-like properties in some texts of the medieval Euclidean tradition. It starts with an overview of propositions in the *Elements* that, retrospectively seen, embody treatments of distributivity-related properties of multiplication over addition or subtraction. The main sections of the article discuss significant changes underwent by these basic conceptions in some influential medieval mathematical treatises, and the concomitant changes arising in the treatment of distributivity. The texts discussed comprise contributions by al-Nayrīzī, Abu Kāmil, the *Liber Mahameleth*, Fibonacci, Jordanus Nemorarius, Campanus de Novara, and Gersonides. The perspective afforded by the use of distributivity in its various manifestations gives rise to some additional insights concerning medieval attitudes towards the questions of what are numbers, what are their basic defining properties, and what is the right way to provide clear foundations for arithmetic as a solid mathematical field of knowledge.

2. *Introduction*

In the present article I analyze the relationship between geometry and arithmetic in the medieval Euclidean traditions, as seen from the perspective of "distributivity-like" situations arising in various kinds of texts. My analysis here is best seen as complementary to a previous one, where I discussed the ways in which results from Book II of Euclid's *Elements* were treated by medieval authors, commentators and translators. (Corry 2013; hereafter referred as [LC1]). I started there by analyzing Euclid's own version of Book II, while fully endorsing the criticism put forward in (Unguru 1975) vis-à-vis the so-called "geometric algebra" interpretation of Greek mathematics. At the same time, however, I stressed the inadequacy of an unqualified use of those terms (geometry, arithmetic, algebra) as if they referred to bodies of knowledge that are easily recognizable in ahistorical terms, and traces of which we may either clearly find or fail to find in any given ancient text. To the contrary, I discussed the ways in which new kinds of ideas gradually appeared in the medieval versions of Book II and that transformed not only the formulations and the proofs of the propositions, but also the very borderlines and interactions among mathematical disciplinary matrices.

"Distributivity-like" properties of multiplication over addition appear in interesting ways in medieval texts and as such they deserve a focused treatment. However, I do not mean to imply in my account below that, in any of the texts discussed here, we find a general, clearly formulated idea of "distributivity" as a fundamental, widely acknowledged, kind of property underlying the relationship between two basic and also well-defined operations, "product" and "addition", and for which there is are specific manifestations in different contexts. The reader should keep this important remark in mind, and understand their use in their own, ever-changing historical and disciplinary contexts.

The ideas discussed here in relation with “distributivity-like” properties developed and consolidated as part of a long common process of interaction involving various kinds of ideas: product, addition, number, magnitudes. I think that it is historically rewarding to look at these earlier developments from a common perspective that involves a broad idea of “distributive-like” properties. Accordingly, then, the term is used here as a general, non-essentialist label that allows a common reference to various kinds of results that bear important similarities, rather than as an assertion that this was a clearly conceived, general idea specifically applied in particular cases.

I shall be referring here only to medieval texts already discussed in [LC1]. Hence I will focus only on the mathematical issues directly connected with distributivity-like properties themselves. I shall not repeat here the historical descriptions and broader contextual considerations already discussed in [LC1]. Those are highly important issues and they are also relevant to some of the questions discussed here. The interested reader is referred to [LC1] for further details.

In the first part of the article I give a preliminary overview of some propositions of Euclid’s *Elements* that, retrospectively seen, embody results that we can associate nowadays with a more general idea of distributivity. This overview is not intended as a detailed historical analysis of a general concept of distributivity that we can putatively attribute to Euclid. Nor do I wish to analyze here the actual, underlying conceptions behind Euclid’s uses of additions and products of numbers or of magnitudes, and certainly not to elucidate questions related to their historical roots. Rather, my intention is just to map out and to cursorily discuss a specific set of Euclidean propositions from books II, V, and VII, that, later on, *medieval authors* referred to or relied upon in their own works, when discussing the role of distributivity-like rules. A main focus of interest will concern the way in which results and arguments that in Euclid's *Elements* appear in the clearly separate contexts of numbers, magnitudes and proportions, started to appear in medieval texts intermingled with one other.

The medieval texts discussed in the main section of the article comprise contributions by al-Nayrīzī, Abu Kāmil, the *Liber Mahameleth*, Fibonacci, Jordanus Nemorarius, Campanus de Novara, and Gersonides. The prism of distributivity provides some interesting insights into the medieval conceptions of what are numbers, what are their basic defining properties, and what is the correct way to provide a solid foundation for this mathematical discipline.

Throughout the text, I render in modern symbolic terms some of the results discussed. Unguru's 1975 article targeted such renderings as part of his criticism of the "geometric algebra" interpretation. The fact that I have rendered some results symbolically is not meant to indicate that in my view this is the correct way to understand them in their historical setting. Moreover, besides this general disclaimer, which should apply to the entire article, I have added in some places indications about specific shortcomings of interpreting a given result symbolically. Still, in spite of these shortcomings the symbolic rendering sometimes serves as convenient shorthand to nail down a specific part in an argument or to cross-refer among results appearing the various texts. I stress once again that, in general, I am not using this symbolic rendering as an interpretation of what either Euclid or the other authors discussed had in mind when formulating these propositions.

As a final, preliminary note, I call attention to the fact that, as in [LC1], I have followed the convention of writing some paragraphs using a different font. This is meant as an indication to the reader that these paragraphs comprise purely technical details of some of the proofs, and that they are intended as evidence in support of the general claims made in the corresponding sections. The paragraphs may be read with due technical attention, or they may be skipped at least temporarily without thereby missing the general line of argumentation.

3. *Distributivity-like Results in Euclid's* Elements

To the extent that distributivity-like properties are discussed as an issue of inherent, focused interest in Euclid's *Elements*, that happens in Book II. Inasmuch as we can consider rectangle formation as a kind of multiplication, and the concatenation of rectangles having one side of equal length as addition, the first four propositions of Book II can be said to embody distributivity-related ideas of the former over the latter Proposition II.1, as is well known, formulates the following general property:[1]

> **II.1:** If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

The main step in the argument of the proof relies on a proposition from Book I, I.34, while making reference to the following diagram, where in the rectangle *BGHC*, the side *BG* equals the given line *A*, and the line *BC* is divided into sub-segments:

[1] All the quotations of the *Elements* in this section, as well as the accompanying diagrams, are taken from (Heath 1956 [1908]). For an enlightening discussion of the issues involved in the use of diagrams related to ancient Greek sources, see (Saito and Sidoli 2012).

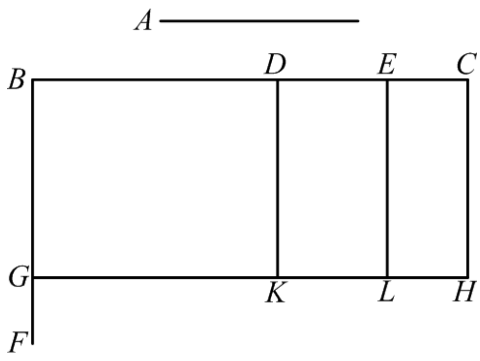


Figure 1: Euclid's *Elements* II.1

The said proposition I.34 is used to assert that, since *BK* is by construction a rectangle, then *DK* equals *BG* and hence equals *A*. A repeated application of this argument allows concatenating the three resulting rectangles into a single, larger one, and thus to complete the proof.

Propositions II.2-II.3 may be seen as particular cases of II.1:

> **II.2:** If a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole.

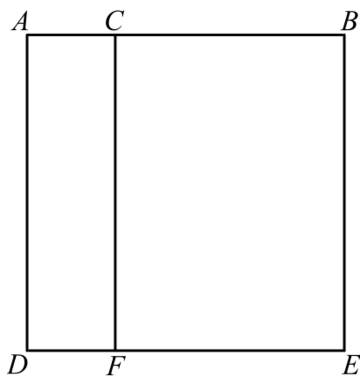


Figure 2: Euclid's *Elements* II.2

> **II.3:** If a straight line is cut at random, then the rectangle contained by the whole and one of the segments equals the sum of the rectangle contained by the segments and the square on the aforesaid segment.

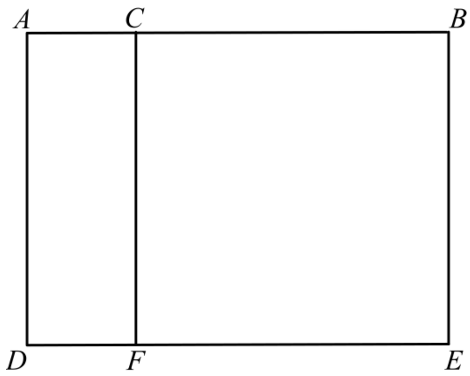


Figure 3: Euclid's *Elements* II.3

In spite of their appearance as particular cases of II.1, Euclid did not prove them by straightforward application of the latter. Rather, the two proofs he offered essentially recapitulate the argument of II.1, but now applied to the particular cases in point. This reflects a more general feature typical of the proofs of the first ten propositions of Book II, namely that none of them relies on a previous one of the same book. Rather they are all proved by directly relying on propositions of Book I alone. In particular this is also the case of proposition II.4, which can as well be seen—mathematically speaking—as a particular application of II.1. It embodies a distributivity-like property of square-formation over division of a line into two parts, which—like the previous two propositions—is not proved in the *Elements* based on II.1, but rather by direct application of I.34. Its formulation and diagram are as follows:

> **II.4:** If a line is cut at random, then the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

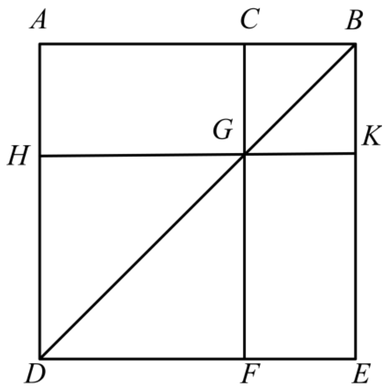


Figure 4: Euclid's *Elements* II.4

Book V

The first six propositions of Book V can be seen as a self-contained, comprehensive discussion of results concerning equimultiplicity of continuous magnitudes (Acerbi 2003). As we will see below, medieval treatises where distributivity-like properties are discussed do mix arguments taken from Book II with those taken from these specific propositions of Book V. For this reason I have chosen to include them here in this preliminary discussion. Within the *Elements*, these propositions provide basic results that are used to develop in full the Eudoxian theory of ratios and proportions later in the book. Ratios and proportions as such, however, are not yet mentioned in these six propositions. They deal only with magnitudes and their multiplicities.

These are the enunciations of the six propositions:

> **V.1:** If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitude equal in multitude, then, whatever multiple one of the magnitude is of one, that multiple also will all be of all.
>
> **V.2:** If a first magnitude be the same multiple of a second that a third is of a fourth, and a fifth also be the same multiple of the second that a sixth is of the

fourth, the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth.

V.3: If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third, then also *ex aequali* the magnitudes taken will be equimultiples respectively, the one of the second and the other of the fourth.

V.4: If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

V.5: If a magnitude be the same multiple of a magnitude that a part subtracted is of a part subtracted, the remainder will also be the same multiple of the remainder that the whole is of the whole.

V.6: If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them are equimultiples of the same, the remainders also are either equal to the same or equimultiples of them.

Heath' symbolical rendering of these proportions sets the stage for possibly interpreting them as general statements of distributivity laws. It involves the following expressions:

V.1: $m \cdot a + m \cdot b + m \cdot c + \cdots = m \cdot (a + b + c + \cdots)$.

V.2: $m \cdot a + n \cdot a$ is the same multiple of a as $m \cdot b + n \cdot b$ is of b.

Further, says Heath, from the proof we learn that $m \cdot a + n \cdot a + p \cdot a + \cdots = (m + n + p + \cdots) \cdot a$.

V.3: $(m \cdot n) \cdot a = m \cdot (n \cdot a)$.

V.4: If $a:b :: c:d$, then $m \cdot a : n \cdot b :: m \cdot c : n \cdot d$.

V.5: $m \cdot a - m \cdot b = m \cdot (a - b)$.

V.6: If $n < m$, $m \cdot a - n \cdot a$ is the same multiple of a as $m \cdot b - n \cdot b$ is of b.

But beyond the general historiographical issues raised by the use of this kind of symbolic translation, in this particular case, Heath's rendering is potentially misleading in a more specific sense. Indeed, the "multiple" of a given magnitude M is not, for Euclid, the outcome, $n \cdot M$, of a binary operation, namely, multiplying a given number n by the magnitude M. Nor is it the result, $M + M + M + M + \cdots + M$, of successive steps of binary additions of the magnitude to itself n times. Rather, it is more of an "accumulation" or of a "gathering together" of a *multitude* of instances of the said magnitude: $M, M, M, M, \cdots M$. Also, *V.2–V.3* above involve the particular difficulty that nothing like the coefficient multipliers m,n do appear in Euclid's text (Taisbak 1971, 65).

Heath's rendering also does not reflect (except in the case of V.4) the "if … then" style of formulation of the propositions. Thus, a symbolic rendering that would remain closer to Euclid in at least this formal and important respect could be the following:

V.1': If $a' = m \cdot a$, $b' = m \cdot b$, $c' = m \cdot c, \cdots$ then $a' + b' + c' \cdots = m \cdot (a + b + c \cdots)$

V.2': If $a = m \cdot b$, $c = m \cdot d$ and $e = n \cdot b$, $f = n \cdot d$, then, while $a + e = p \cdot b$, we also have $c + f = p \cdot d$. In this symbolic rendering one sees that $p = m + n$.

V.3': If $a = m \cdot b$, $c = m \cdot d$ and $e = n \cdot a$, $f = n \cdot b$, then, $e = p \cdot a$, $f = p \cdot d$. In this symbolic rendering one sees that $p = m \cdot n$.

V.4': If $a:b :: c:d$, then $m \cdot a : n \cdot b :: m \cdot c : n \cdot d$.

V.5': If $a = m \cdot b$, $c = m \cdot d$, then $a - c = m \cdot (b - d)$.

V.6: If $a = m \cdot b$, $c = m \cdot d$ and $e = n \cdot b$, $f = n \cdot d$, then while $a - e = p \cdot b$, we also have $c - f = p \cdot d$. In this symbolic rendering one sees that $p = m - n$.

Notice a point of particular interest in this regard concerning V.6. Euclid mentions separately the possibility that the remainders be "equal to the same". Symbolically this corresponds to the case $m = n + 1$ and there is no need to speak about it separately. But for Euclid, this is indeed a separate case, namely one in which we do not have a *multitude* of instances of the magnitude but just one instance. Hence the need to mention it separately.

The proofs of the first six propositions in Book V are based on various way of counting the multitudes of the magnitudes involved in each case. Of special interest for us are the proofs of V.1 and V.2 because they will resurface in modified ways in the medieval texts discussed below. As for V.1, the proof is accompanied by the following diagram:

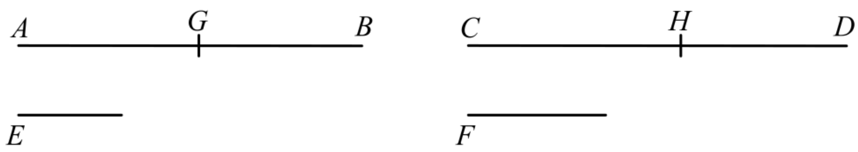


Figure 5: Euclid's *Elements* V.1

Here *AB* and *CD* represent equimultiples of *E, F* respectively. The proposition states that *AB, CD* taken together is that same equimultiple of *E, F* taken together.

The proof follows closely the diagram and explains that *AB* is divided into (two) lesser magnitudes *AG, GB* (each equal to *E*) while *CD* is divided into the same number (two) of lesser magnitudes *CH, DH* (each equal to *F*). Now, the lesser magnitudes of the two kinds are joined into pairs (each equal to *E, F*) that can be used to measure the sum *AB, CD*. And the amount of such pairs turns out to be equal to the times (two) that each of the two lesser magnitudes measured both *AB* and *CD* at the beginning of the proof. In Euclid's words: "since *AG* is equal to *E* and *CH* to *F*, therefore *AG* is equal to *E*, and *AG*, *CH* to *E*, *F*. For the same reason *GB* is equal to *E*, and *GB*, *HD* to *E*, *F*. Therefore as many magnitudes as there are in *AB* equal to *E*, so many also are there in *AB*, *CD*, equal to *E*, *F*".

Again for ease of reference below, rendering the argument of the proof in modern symbols, while at the same time generalizing for more than two summands for the first and the second magnitudes, yield the following:

(a.1) $AB = E + E + E + \cdots + E$ (n times)

(a.2) $CD = F + F + F + \cdots + F$ (n times)

(a.3) $AB + \mathrm{CD} = (E + E + E + \cdots + E) + (F + F + F + \cdots + F)$

(a.4) $AB + \mathrm{CD} = (E + F) + (E + F) + \cdots + (E + F)$ (n times)

Euclid lacked a flexible symbolic language that could allow him to formulate the proof in all of its generality. Still, the generality of the argument seems to be compromised neither in terms of the multiplicity involved ("two" in the proof) nor of the number of magnitudes that added (here: *E* and *F*). In this particular sense, the generality implied in the symbolic rendering ***V.1*** does not seem to go beyond what Euclid stated in the enunciation and the proved in the proof. His main trick lies in pairing the measuring magnitudes and then counting the ensuing pairs.

Let us consider now the proof of V.2, whose diagram is the following:

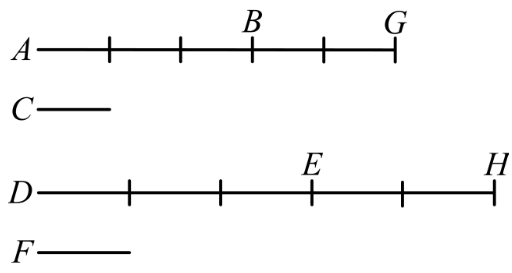


Figure 6: Euclid's *Elements* V.2

Here *AB* is the same multiple of *C* that *DE* is of *F*, and likewise, *BG* is the same multiple of *C* that *EH* is of *F*. The proposition states that *AG* is the same multiple of *C* that *DH* is of F. Notice that unlike in V.1, *C* and *F* need not be magnitudes of the same kind, and the proposition does not involve in any way gathering together the first or second magnitudes with the third or fourth, respectively. Rather, the proposition only compares two separate relationships between pairs, each pair comprising magnitudes of the same kind.

> The starting point of the proof is that as many magnitudes are in *AB* that are equal to *C*, so many are also in *DE* that are equal to *F* (and the same goes for *BG* and *EH*). It is a nice feat, I think, that Euclid attempted to infuse some kind of generality to the argument by taking three and two as the multiplicities (rather than, for example, two and two). Now the two multiplicities are gathered together with each other separately and as a result "as many as there are in the whole *AG* equal to *C*, so many are there in the whole *DH* equal to *F*."

The cases involving subtraction are considered in V.5 and V.6, which are parallel to V.1 and V.2. The proofs of the second pair are somewhat different from those of their counterparts in the first one. One point of interest concerning V.5 can be mentioned with reference to the accompanying diagram, which is the following:

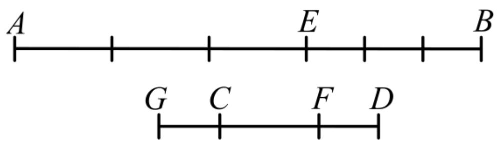


Figure 7: Euclid's *Elements* V.5

> Here the magnitude *AB* is the same multiple of *CD* than *AE* is of *CF*. *AE* is subtracted from *AB* to obtain *EB*, while *CF* is subtracted from *CD* to obtain *CF*. The proposition states that the remainder *EB* is also the same multiple of *FD* as *AB* is of *CD*. The segment *CG*, not mentioned in the enunciation, is constructed as part of the proof, so that *EB* be the same multiple of *CG* that *AE* is of *CF*. Symbolically: if $AE = n \cdot CF$, then $EB = n \cdot CG$.

The interesting point in the proof is Euclid's assumption of the existence of *CG*. Indeed this is tantamount to assuming that given any magnitude, here *EB*, we can divide it into as many equal parts as we wish. In Euclid's own diagram, for instance, *FD* is a third of *AE*, so that *CG* needs to be made a third of *EB*. This would seem to present no problem if we were dealing with segments, as in the diagram, but in fact it is only in VI.9, that Euclid proves that such a fourth line exists. In the case of areas or volumes, it would seem more difficult to figure out how this fourth magnitude is found. And, much worse, in the case of angles or arcs, it may turn out to be impossible (or at least impossible with rule-and-compass methods). In his comments to this proposition, Heath mentioned alternative proofs that were suggested by later authors and that bypass this problem (Heath 1956 [1908], Vol. 2, 146. See also Mueller 1981, 122).

A further, interesting point to notice in relation with these six propositions of Book V is that their diagrams are all half way between the geometric (typical of proofs in Book II), and the purely arithmetic (such as in Book VII). Thus, on the one hand, the magnitudes are represented in the diagrams by lines, while they are meant as magnitudes of any kind, including areas, angles or volumes (but not numbers). On the other hand, these lines are not parts of constructions (like in the geometric proofs) but are just indicative of the magnitudes involved. As in the arithmetic type of proofs, the only way in which they are manipulated upon is that they are added to, or subtracted from, each other (Mueller 1981, 122). Again, this will appear as an important point below, because, as it is well known, Book VII develops a theory of proportions for natural numbers, which runs parallel to that of Book V, and in some medieval texts we find discussions about the need to follow Euclid in maintaining this separation.

Book VII

Also in Book VII of the *Elements* we find propositions that involve distributivity-like properties. They focus on the question of "being a part" or of "being parts" of a given number and, as in Book V, they explore the issue equimultiplicity. We have two pairs of propositions, VII.5-VII.6 and VII.7-VII.8, that appear as parallel, respectively, to V.1 and V.5. They are formulated as follows:

> **VII. 5:** If a number be a part of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the one.
>
> **VII. 6:** If a number be parts of a number, and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one.
>
> **VII. 7:** If a number be that part of a number, which a number subtracted is of a number subtracted, the remainder will also be the same part of the remainder that the whole is of the whole.
>
> **VII. 8:** If a number be the same parts of a number that a number subtracted is of a

number subtracted, the remainder will also be the same parts of the remainder that the whole is of the whole.

Also in this case, the symbolical rendering suggested by Heath stresses very much a possible retrospective interpretation of these propositions as statements of distributivity of multiplication of numbers over their addition:[2]

VII.5: $\frac{1}{n}a + \frac{1}{n}b = \frac{1}{n}(a+b)$

VII.6: $\frac{m}{n}a + \frac{m}{n}b = \frac{m}{n}(a+b)$

VII.7: $\frac{1}{n}a - \frac{1}{n}b = \frac{1}{n}(a-b)$

VII.8: $\frac{m}{n}a - \frac{m}{n}b = \frac{m}{n}(a-b)$

But this rendering raises, beyond the historiographical issues already mentioned for any symbolic rendering of this kind, the additional problem that in Euclid's arithmetic there is nothing like the fraction $\frac{1}{n}$. Perhaps a closer symbolic approximation—not without its own limitations as a faithful rendering of Euclid—could be the following:

VII.5: If $a = n \cdot b$ and $c = n \cdot d$, then $a + c = n \cdot (b + d)$.

VII.6: If $m \cdot a = n \cdot b$ and $m \cdot c = n \cdot d$, then $m \cdot (a + c) = n \cdot (b + d)$.

VII.7: If $a = n \cdot b$ and $c = n \cdot d$, then $a - c = n \cdot (b - d)$.

VII.8: If $m \cdot a = n \cdot b$ and $m \cdot c = n \cdot d$, then $m \cdot (a - c) = n \cdot (b - d)$.

In strict mathematical terms this latter rendering is equivalent to Heath's but it expresses more closely the spirit of the original. For one thing, contrary to what is

[2] A similar, though not identical, rendering appears in (Itard 1961, 90-97). (Taisbak 1971, 40 ff.) gives a completely different kind of symbolic rendering, conceived with the specific purpose in mind of avoiding any possible historical inaccuracy incurred by the use of modern algebraic symbolism. It is well beyond the intended scope of the present article to follow any kind of symbolic approach close to that of Taisbak. Still, it is interesting that, following his own point of view, Taisbak states explicitly (p. 43) that VII.5 and VII.6 "can be interpreted as the *Distributive Law*".

the case with Heath's rendering, this one retains the "if .. then" format of Euclid's original formulation. For another thing, in Heath's rendering, the four statements are just different instances of the same general rule of distributivity. This raises the question of why Euclid would have chosen to state and prove them as separate results. Obviously, the straightforward equivalence between the four becomes possible only when written retrospectively in this way. This misleading interpretation is not fully absent from the alternative rendering I just suggested, but it is at least somehow mitigated. I think it does provide a justification for seeing them as part of a family of related, distributive-like properties, without thereby having to assume the existence of a clear-cut, general idea of distributivity.

Euclid's four proofs are quite similar to each other and they are based on a rather straightforward counting of units. We can see the main argument through the example of VII.5, which is accompanied by the following diagram:

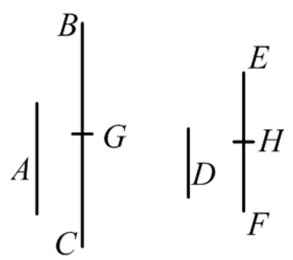


Figure 8: Euclid's *Elements* VII.5

Here *A* is a part of *BC*, and *D* is the same part of another *EF* that *A* is of *BC*. The proposition then states that the sum of *A*, *D* is also the same part of the sum *BC*, *EF* as *A* is of *BC*. The core of the argument recapitulates that of V.1.[3]

Since we shall need this below, it is pertinent to say also a word about the proof of VII.6, which is accompanied by the following diagram:

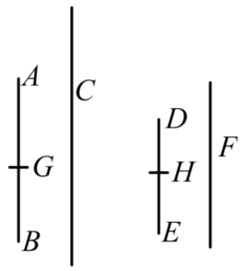


Figure 9: Euclid's *Elements* VII.6

In the proof, *AC*, *GB* represent parts of *C* that taken together make the number *AB*. Likewise, *DH*, *HE* represent parts of *F* that taken together make the number *DE*. As many parts of *C* as taken together make *AB* so many parts of *F* taken together make *DE*. Hence, one can apply VII.5 and conclude that "whatever part *AG* is of *C*, the same part also is the sum of *AG*, *DH* of the sum of *C*, *F*." The same is then concluded for *GB* vis-à-vis *C*, and for the sum *GB*, *HE* vis-à-vis the sum of *C*,*F*. And finally, "whatever parts *AB* is of *C*, the same parts also is the sum of *AB*, *DE* of the sum of *C*, *F*."

Notice that for lack of a more flexible language, Euclid took *two* parts in the proof to mean "an arbitrary number of parts." In the argument, he referred to each of these parts separately, *AG* and *GB*, and then moved to the conclusion for the sum. But he did not say explicitly that this conclusion requires that the procedure can indeed be validly applied to the case in which three, four, or any number of parts of *C* other than two is added. It is quite clear, nevertheless, that the generality of

[3] According to (Taisbak 1971, 41), the details of this proof indicate that Euclid implicitly takes for granted the associative and commutative laws for the addition of numbers, "implied as they are in his definitions of number".

the argument, in this regard, is not compromised. Notice also that it is not known how many of parts such as *AG* taken together make *C*, but we do know that as many as they are, *F* is made of as many parts, each being like *DH*. Obviously then, the generality of this part of the argument is not compromised in any sense, because we do not even need to now into how many parts *C* or *F* are divided.

Referring to the symbolic rendering in *VII.6*, $\frac{m}{n}a + \frac{m}{n}b = \frac{m}{n}(a+b)$, we can say that Euclid's argument is clearly valid for any value of n, because n is actually absent in the argument. At the same time, the argument is presented in detail only for $m = 2$, and it is left implicit that it can be generalized to any value of m, though it is not obvious how this could be done in the language typically available to Euclid.[4]

The proofs of VII.7-VII.8 amount to not much than similar counting of units as in the previous two propositions, and both of them reduce their argument to a situation where VII.5 can be directly applied.

Using Distributivity-Like Properties in the Elements

Let us take a look now at the ways in which the various distributivity-like propositions discussed above are put to use in the *Elements*. It is interesting to notice that, taken together, they are used in crucial places for many more elaborate propositions in the treatise. Starting with Book II, the following diagram shows how the first ten propositions are used:

[4] See (Itard 1961, 93-97) for what he considers to be a problematic aspect of Euclid's proofs for VII.6 and VII.8. See (Taisbak 1971, 42-48) for additional, but quite different kinds of comments.

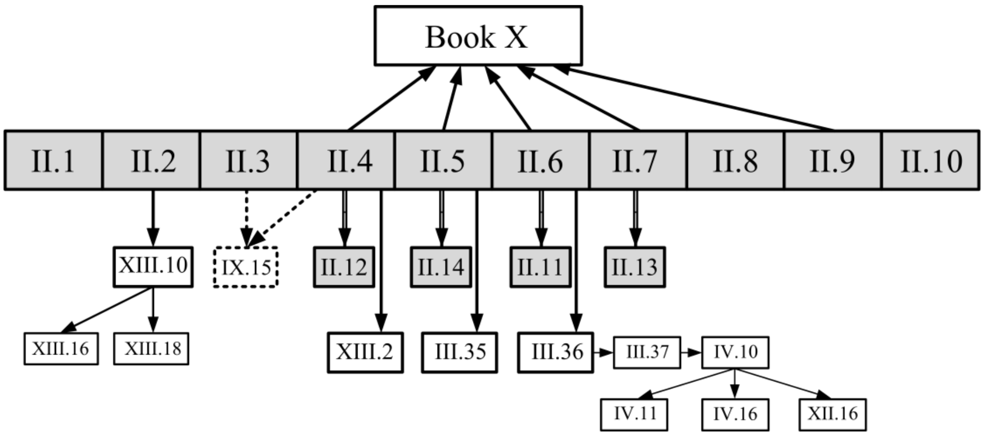


Figure 10: Deductive dependence of Distributive-like results in Euclid's *Elements*

As it is well know, and as shown in the diagram, a main application of the propositions appearing in the first part of the book is in the proofs of II.11 to II.14. The latter embody results that are mathematically significant in themselves (e.g., cutting a segment in mean and extreme ration in II.11 and constructing a square equal to a given rectilinear figure in II.14). These two also have important applications in later parts of the *Elements* (such as in IV.10, which is needed for constructing the regular pentagon, and which relies on II.11). But from the diagram, one also readily notices that Book X appears as a main focus of application of these ten results, and specifically of II.4-II.9. Taken together with the applications in proofs in Books III, IV, XII XIII, we can say that these distributivity-like properties of area-formation do play an important role in the general economy of Euclid's treatise. A case of special interest for our discussion here is that of IX.15, in which proof, *arithmetic* versions of II.3-II.4 are used. I shall further comment on this issue right below.

As already pointed out, Euclid's basic approach in developing the first ten results of Book II was to prove each proposition on the basis of results of Book I alone. Technically speaking, however, it would be enough to prove II.1, and then all the rest could be proved by relying on this one proposition alone and without further recourse to Book I. This is, as we shall see below, the approach that Heron followed in his version of Book II. One can only speculate about the reasons behind Euclid's specific choice. Together with the issue of "geometric algebra" and the application of the propositions of Book II to prove additional propositions in the treatise, historians have discussed the possible significance of the approach followed by Euclid in his proofs. Such discussions concern an assessment of the role of Book II as a whole—and of some of its individual propositions separately—within the general economy of the *Elements*. Ian Mueller, for instance, found the evidence inconclusive to come up with such an assessment (Mueller 1981, 301-302):

> If one accepts II,11-14 as a goal of book II, one has an explanation for the presence of II,4-7, which are used in their proofs. … As far as I am able to determine, there is nothing in the *Elements* themselves which makes the algebraic interpretation of these propositions more natural than the straightforward geometric one. On the other hand, the minimal use of II,1-3, 8-10, together with the generally loose connection between book II and books X and XIII, makes it difficult to feel confident about book II. … What unites book II is the methods employed: the addition and subtraction of rectangles and squares to prove equalities and the construction of rectilinear areas satisfying given conditions. 1-3 and 8-10 are also applications of these methods; but why Euclid should choose to prove exactly those propositions does not seem to be explicable.

Christian Marinus Taisbak, in turn (Taisbak 1993, 30), speculated about a possible explanation that builds on some pre-Euclidean, ancient arithmetic traditions. In those traditions, propositions similar to II.5, II.6, II.10 and II.11 appear in a natural way. Taisbak added that II.1 should be considered, under this view, to have been added by

Euclid himself as a generalization of other propositions presented in Book II. He regretted, moreover, that Euclid "did not tell us explicitly what is the meaning of it all, particularly what II.5 and II.6 are good for, although such knowledge is presupposed in Book X from prop. 17 onward."

Now, one may wonder to what extent, if at all, the medieval authors to be discussed below ever asked themselves *historical* questions of this kind or what was their view on issues related to these ones. We do know, as will be seen below, that they occasionally came up with their own alternative *mathematical* approach to the proofs and to the connection among the various propositions. But in order to complete the picture of this part of the discussion, it is also important to stress that in several places in the *Elements* we find proofs where distributivity-like properties of area formation over addition are implicit used without any explicit comment or further justification. The typical case in point is in the proof of I.47, where two rectangles with a common side are added to create a certain square (see also [LC1]). This can be seen in the diagram of the proof, which is the following:

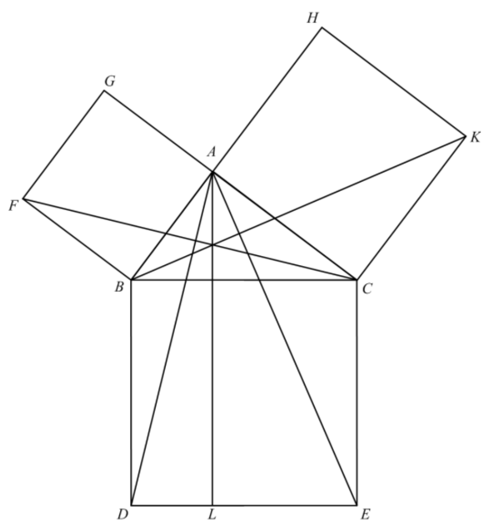


Figure 11: Euclid's *Elements* I.47

A crucial step in the proof is that

$$\mathrm{Sq}(BC) = \mathrm{R}(BD,DL) + \mathrm{R}(CE,LE),$$

a step that is simply indicated with reference to the diagram. But this is precisely the kind of situation that is handled, and duly proved in II.1-II.3. How can we then explain that Euclid takes the step in Book I without further comments but finds it necessary to prove the same point in Book II? This issue has been discussed by Ken Saito (Saito 2004 [1985]), who called attention to the fact that the two rectangles added in the proof of I.47 are explicitly drawn (or "visible") in the diagram. All rectangles appearing in the proofs of II.1-II.3, on the contrary, do not arise as part of an explicit geometric construction in the corresponding diagrams (Saito calls them "invisible"). Justification such as provided by II.1 is necessary, Saito convincingly argues, because the rectangle to which the statement refers is invisible. Moreover, the crucial point of the proofs of II.1-II.3 is that the "invisible" rectangles referred to in the proposition are made "visible", so that this kind of "distributivity of area-formation", which is self-evident for "visible" rectangles, can be used.

Seen from this perspective, the first ten propositions of Book II are lemmas that are introduced beforehand, in preparation for later use in specific geometric situations. In those situations, it is possible to invoke the lemma without having to explicitly show in the diagram the relevant construction, which thus remains "invisible". We find a good example of this in XIII.10, which is the only place where II.2 is explicitly used. The proposition states than in an equilateral pentagon inscribed in a circle, the square on the side of the pentagon equals the sum of the squares on the sides of the hexagon and the decagon that are inscribed in the same circle. The diagram is the following:

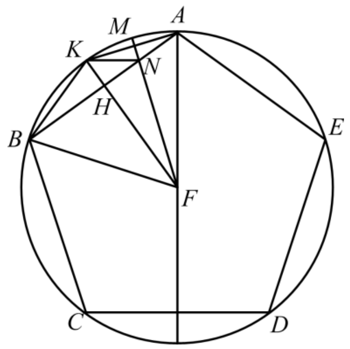


Figure 12: Euclid's *Elements* XIII.10

The proof is relatively complicated and we do not need to see all its details here (but see Taisbak 1999). I just want to focus on the fact that one of the concluding steps of the proof requires that

$$\text{Sq}(BA) = \text{R}(AB,BN) + \text{R}(AB,AN).$$

The step is justified on the basis of II.2, and thus the proof is completed without drawing any of these invisible figures explicitly. This also help explains why Euclid proved separately II.2-II.3, in spite of their appearance as no more than particular cases of II.1, because each of them needs to be ready for the specific situation in which it is going to be used (Saito 2004 [1985], 164).

An additional focal point of interest that deserves special attention here is the proof of IX.15. Several stages of the argument rely on what can be seen as arithmetic versions of II.3 and II.4 (actually II.3 is not applied anywhere else in the *Elements*). We have seen that, because Euclid consistently adhered to the principle of separating realms throughout the treatise, separate versions are found of what can retrospectively be seen as similar distributivity-like results. Here, however, we have an interesting case of transgressing that principle.[5] The proposition in question is stated as follows:

> **IX.15**: If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number.

The accompanying figure is the following:

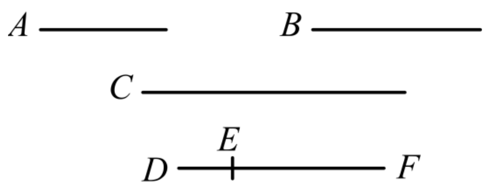


Figure 13: Euclid's *Elements* IX.15

Here A, B, C are the three given numbers in continued proportion. Proposition VIII.2 warranties the existence of two numbers, DE, EF such that $A = DE^2$; $B = DE \cdot EF$; and $C = EF^2$. In addition, by VIII.22, DE, EF are mutually prime. Now, along the proof, Euclid considers several *products* involving DF, DE, EF and their squares. Using various propositions of Book VII (22, 24, 25), he can establish certain relations of mutual primality among them. But in other places he also needs to consider cases where the numbers and their squares are *added* to one another, and it so happens that in Books VII and VIII, there is only *one* proposition (VII.28) that handles cases of adding mutually prime numbers. Euclid indeed invokes this proposition in order to prove that DF is relatively prime to both DE and EF. But in the other stages in the proof he deals with additions not covered for VII.28, and that can be interpreted as arithmetic cases of the situation covered (for the geometric case) by II.3 and II.4. Thus, for instance:

[5] See (Mueller 1981, 108 ff.) for a broader discussion of Euclid's use of geometric arguments and analogies in arithmetical contexts. For example, Mueller indicates (p. 108) that in Book X, Euclid proves two *lemmatas* while invoking an arithmetic analogue of II.6.

- The product of *FD*, *DE* is the square on *DE* together with the product of *DE*, *EF*.
- The squares on *DE, EF* together with twice the product of *DE*, *EF* are equal to the square on *DF*.

Now, in preparation for this proof, Euclid could have conceivably formulated and proved the purely arithmetic propositions needed here, but for some reason he declined to do so and preferred to deviate from his self-imposed, strict separation of domains. Again, we can only speculate about the reason for this choice, but it is interesting to mention that in the medieval texts discussed below we shall encounter an interesting twist to this peculiar situation. Al-Nayrīzī in the 10th century, and then Campanus in the 13th century, formulated a generalized version of IX.15 and added to it original "commentaries" most of which happen arithmetic versions of results from Book II. This is one of the most interesting issues that I discuss in detail below.

Let us consider now the way in which the six propositions of Book V discussed above are used in the *Elements*. This is schematically represented in the following diagram:

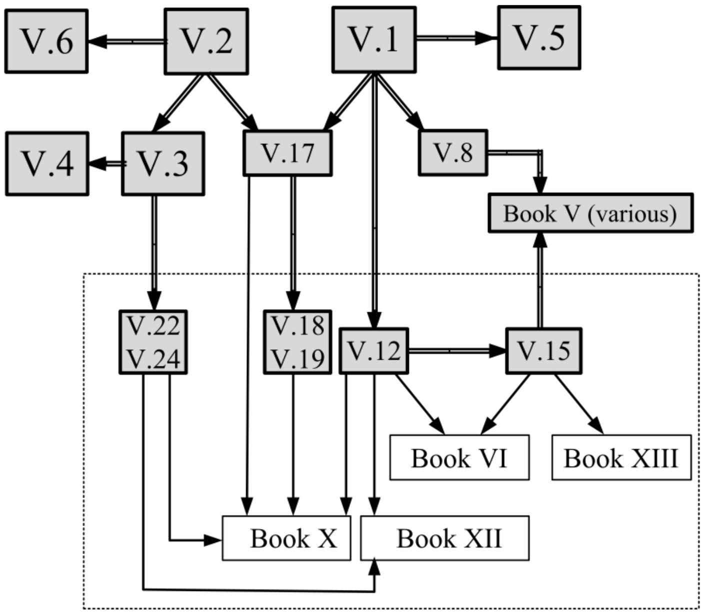


Figure 14: Deductive dependence for distributive-like results in Euclid's *Elements*

In the lower dotted square I have indicated most, but not all, of the results that can be derived indirectly from the six propositions that interest us here. Still, the picture is quite clear and interesting. While V.1 and V.2 are used *crucially* in some of the later proofs in the book, V.3 is used only in the proof of V.4, whereas V.5, V.6 are not used at all. Also, the derivatives of V.1 and V.2 do play important roles in Books VI, X, XII and XIII.

It is also pertinent to mention in this regard proposition V.24, which may be seen as expressing yet another kind of distributivity-like property of sorts. It reads as follows:

> **V.24:** If a first magnitude have to a second the same ratio as a third has to a fourth, and also a fifth have to the second the same ratio as a sixth to the fourth, the first and fifth added together will have to the second the same ration as the third and sixth have to the fourth.

If we allow ourselves a symbolic rendering of this proposition then, we obtain the following:

V.24**:** If $a{:}c :: d{:}f$, and $b{:}c :: e{:}f$, then $(a + b){:}c :: (d + e){:}f$.

It is important to notice that this proposition appears in the section of book V where Euclid dealt with "composition" of proportions. What I mean by this, are propositions such as V.17 or V.18, whose symbolic renderings are, respectively:

V.17: If $a{:}b :: c{:}d$, then $(a - b){:}b :: (d - e){:}f$

V.18: If $a{:}b :: c{:}d$, then $(a + b){:}b :: (d + e){:}f$.

The proof of V.24 depends crucially on V.18, and it is noteworthy that in his well-known edition of the *Elements*, Robert Simson (1687-1768) indicated that Euclid's proof of V.24 can be easily modified to obtain, based on V.17, a result similar to that of V.24 but involving subtraction. This is something Euclid never did, but as we shall see below, there is at least one medieval text (*Liber Mahameleth*) where this was actually done. What will turn out to be even more interesting, however, is that in that text V.24 is taken to be *the* source of justification for this distributive-like property of the product.

In the *Elements* there is only one other proposition whose proof relies on the main idea of V.24, but it does so with an interesting twist. This is proposition VI.31, which generalizes the Pythagorean theorem by constructing on the sides of a rectangular triangle, not squares but rather any three figures that are "similar and similarly described". Euclid's diagram for VI.31 is the following:

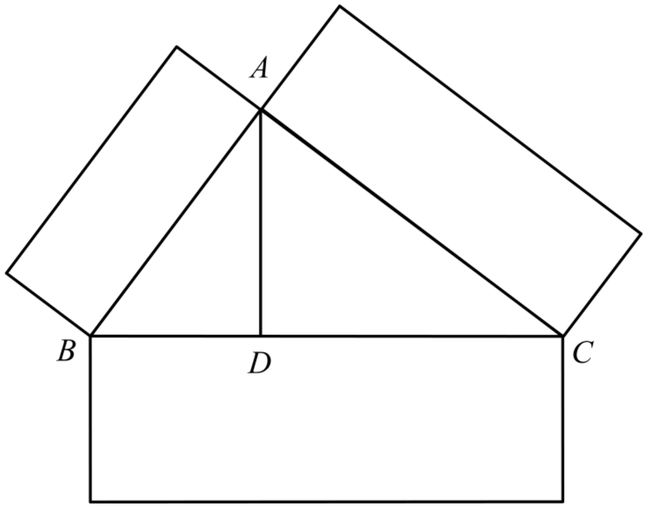


Figure 15: Euclid's *Elements* VI.31

In the proof, Euclid refers to non-specified geometric figures, and not necessarily to rectangles as in the figure. He shows separately that:

(b.1) *CB*:*BD* :: (fig. on *CB*):(fig. on *BA*), and

(b.2) *BC*:*CD* :: (fig. on *BC*):(fig. on *CA*)

and from here he deduces that:

(b.3) *CB*:(*BD* + *CD*) :: (fig. on *CB*):(sum of figs. on *BA* and *AC*).

Completing this latter deduction requires something *close* to V.24, but not exactly the way the proposition is stated. Indeed, in V.24, it is the antecedents that are added in both ratios, and not as here in the proof of VII.31.

Finally, we can take a look at the way in which the six propositions of Book VII discussed above are used in the *Elements*. This is schematically represented in the following diagram:

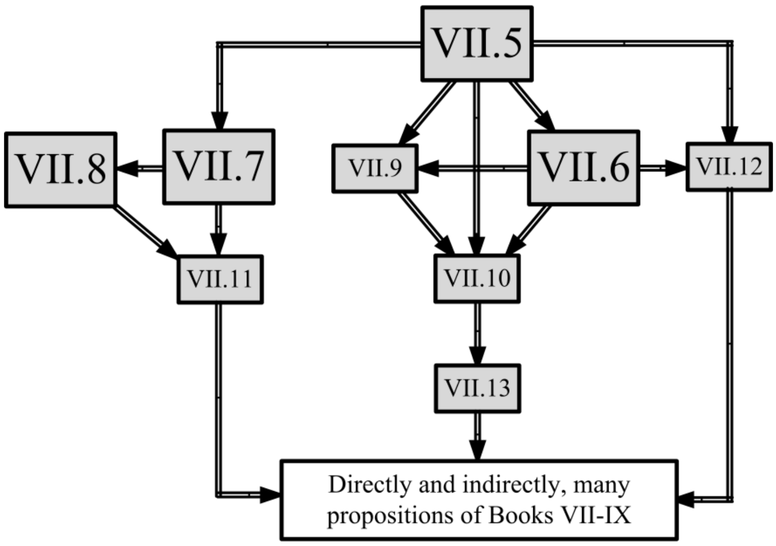


Figure 16: Deductive dependence for distributive-like results in Euclid's *Elements*

In this case we see very clearly that VII.5 (both directly and indirectly via other propositions) provides an important tool that is consistently used for proofs throughout the three arithmetic books of the *Elements*. One important case to mention is that of VII.15-16, which embody the commutativity of the product (and are themselves also used, of course, in the proofs of many other propositions). A second important case is that of VII.11-13, three propositions needed to prove X.35. This latter proposition—in conjunction with several other propositions of Book VII that derive from VII.1—allows Euclid proving IX.36. This is the famous proposition that, in modern terms can be understood as stating that if $2^p - 1$ is a prime number, then $(2^p - 1)\ 2^{p-1}$ is a perfect number.

The above survey may be now summarized as follows:

- Propositions involving distributivity-like properties appear in the *Elements* in three different realms (magnitudes, proportions and arithmetic),
- in each of these realms various cases are treated separately,

- different underlying assumptions, both explicit and implicit, are used in the various proofs, and in the various realms,
- distributivity-like results obtained in the separate realms are put to use in different ways in the overall economy of the *Elements*.

Mathematicians of later historical periods, whose works I discuss below, read the *Elements* from a perspective that comprised changes in some of the basic underlying conceptions. These conceptions concerned the nature of numbers and continuous magnitudes, the relationship among these two different kinds of mathematical entities, and the way they were put to use in various mathematical situations. Among other things, the rather strict separation between the three realms characteristic of Euclid's own approach, was revised and approached differently by the various authors. Moreover, when these mathematicians worked with positive numbers that included, beyond integers, also fractions and even irrational numbers, they could not directly rely on the canonical versions of those propositions that Euclid had proved in his arithmetical books. Book II above all, and to a lesser extent also Book V, provided many results that medieval mathematicians wanted to use in the arithmetic realm, which for them was broader than for Euclid. Their more flexible conception of number directly influenced, as we will see now, the ways in which distributivity-like results were used and justified, while some of the Euclidean propositions were themselves modified accordingly.

4. *Late Antiquity and Islamic Mathematics*

In this section I discuss al-Nayrīzī's additions to the *Elements* that appear in the framework of his report on Heron's *Commentary*. Al-Nayrīzī devoted a more focused attention to distributivity-like results than Euclid had done. I also discuss briefly Abū Kāmil's version of propositions from Book II.

4.1. Heron and al-Nayrīzī

Heron of Alexandria wrote a *Commentary* of the *Elements* at the end of the first century A.D. It contained alternative proofs for several propositions in Book II. In [LC1, 652-654] I contrasted the geometric approaches of Euclid and Heron in their respective proofs, and I described the former as "constructive" and the latter "operational". These differences, however, do not seem to have affected Heron's views on discrete and continuous magnitudes, and the way that they are kept separated throughout the various parts of the *Elements*.

Heron's ideas are known to us via al-Nayrīzī's commentary to the *Elements*, dating from the early tenth century. This is one of the earliest such commentaries to be written in Arabic. Together with his report on Heron's ideas, al-Nayrīzī also added numerical examples of his own that were meant to illustrate each proposition in Book II. The more flexible conception and use of numbers, typical of the Islamic tradition of which al-Nayrīzī was part, certainly helps explain his inclination to go in this direction. But it is important to stress that this kind of numerical interpretation could have hardly been accommodated within the specific approach followed in Euclid's proofs that were, as I said, "constructive". To the contrary, they found a more natural place within Heron's because, while geometrical, they were "operational" [LC1, 661-662].

But al-Nayrīzī's own contribution went much further than just illustrating the propositions of Book II with numerical examples: he also incorporated into the arithmetical books of the *Elements* fully arithmetic versions of some of the propositions of Book II. He did this by adding a section with comments to Book IX. Again we can understand this move against the background of a more flexible conception of number that evolved in Islamicate mathematics, along a less rigid separation of discrete and continuous magnitudes. Less rigid, it must be stressed, but not altogether inexistent. This move, at any rate, had a direct, visible influence on later medieval treatises, and specifically on Jordanus and Campanus as we will see below. Some of the details of Heron's and al-Nayrīzī's proofs are worthy of further examination here.

Heron asserted that II.1 is the only one among the fourteen propositions that "cannot be proved without drawing a total of two lines". For the remaining thirteen propositions, he stated that "it is possible that they be demonstrated with the drawing of one sole line" (Curtze 1989, 89)[6]. There is no report on Heron's argument for II.1, and we may assume that it added nothing to Euclid's original. Al-Nayrīzī's numerical example for II.1, in turn, is embodied in the following diagram (p. 88): [7]

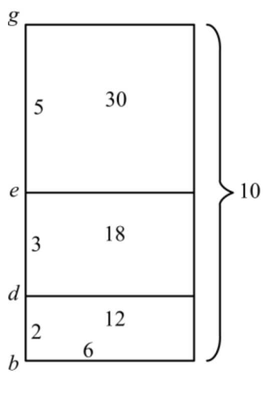


Figure 17: Al-Nayrīzī's diagram for Euclid's *Elements* II.1

[6] Unless otherwise stated, translations from Latin, Hebrew, or German are mine.

[7] In his commentaries to the text of Al-Nayrīzī (Curtze 1898), Curtze added algebraic renderings to each proposition.

Al-Nayrīzī also reported on Heron's proofs for the other propositions in Book II. The most important feature of these proofs is that, unlike Euclid's original proofs, they do not rely on results of Book I. Rather, they rely first on II.1, and then on those other propositions from the same Book II that Heron gradually proved as he went. Thus, for instance, Propositions II.2-II.3 appear here naturally as particular cases of II.1. Then, II.4 appears as directly derivable from II.1, relying also on the other two. We can gain some insight into Heron's approach by looking at his proof for II.4.

As already indicated, Euclid's proof of II.4 is based on I.34, which establishes the equality of the two rectangles *AG*, *GE*. Heron's proof is quite different. For one thing, he did not even draw the full square of Euclid's diagram, but rather, as in the following diagram, only the line *ba*, cut at an arbitrary point *g* (p. 92):

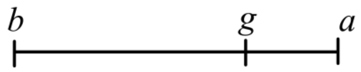


Figure 18: Heron's diagram for Euclid's *Elements* II.4

The proposition states that the square on *ab* equals the sum of the two squares, one on *ag* and one on *gb*, together with twice the area contained by the lines *ag*, *gb*. He did not explicitly show any construction, but it is evident that Heron conceived this proposition, as well as the others in the book, as expressing properties of geometric figures. This same spirit is clearly reflected in the proof itself. His argument for II.4 can be schematically rendered as follows:

(c.1)	By II.2:	Sq(*ab*) = R(*ab*,*ag*) + R(*ab*,*bg*).
(c.2)	But by II.3:	R(*ab*,*ag*) = R(*bg*,*ag*) + Sq(*ag*);
(c.3)	Also by II.3:	R(*ab*,*bg*) = R(*bg*,*ag*) + Sq(*bg*).
(c.4)	Hence:	Sq(*ab*) = Sq(*ag*) + Sq(*bg*) + 2·R(*bg*,*ag*).

In al-Nayrīzī's text, the numerical example is given via the following diagram:

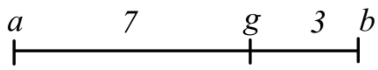


Figure 19: Al-Nayrīzī's diagram for Euclid's *Elements* II.4

Al-Nayrīzī also gave the details of the calculation, namely, that the square on the entire length, 100, equals the sum 49 + 9 + 2·3·7.

But as already mentioned, al-Nayrīzī's own contribution went much further than just illustrating the propositions of Book II with numerical examples: he took the further step of incorporating into the arithmetical books of the *Elements* arithmetic versions of propositions II.1- II.4. These arithmetic versions appear as commentaries added to a result in Book IX (AN-IX.16) that, remarkably enough, does *not* appear in Euclid's original version of the arithmetic sections of the *Elements*.[8] The details of his proofs to these propositions deserve close attention.

The first interesting point to notice in these proofs is that they are typical of those appearing in Euclid's arithmetic proofs. In particular, in the diagrams lines serve to indicate the numbers involved in the proof, but they are not used to produce any relevant *geometric* constructions. Of particular importance is the fact that multiplication is never represented here as area formation.

Al-Nayrīzī's diagram for his version of II.1, as it appears in the corresponding commentary to AN-IX.16, is the following (p. 204):

[8] As a matter of fact, AN-IX.16 is a generalization of Euclid's IX.15. In Euclid's IX.15 three numbers are added, whereas al-Nayrīzī adds an arbitrary number of numbers ("*Si fuerint numeri quotlibet continue proportionales in sua proportione minimi*, …"). Curtze cites the proposition in a footnote (p. 204), without in any way mentioning the discrepancy with the Euclidean original. As we shall see below, Campanus followed al-Nayrīzī's formulation.

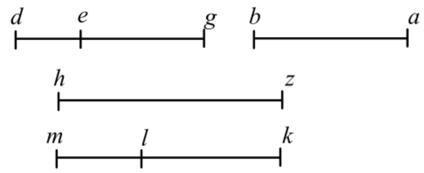


Figure 20: Al-Nayrīzī's diagram for the arithmetic version of Euclid's *Elements* II.1

> The line *hz* represents the product of *ab* by *gd*, whereas *kl* represents the product of *ab* by *ge* and *ml* that of *ab* by *ed*. The proposition states that *hz* equals *km*. The proof proceeds simply by spelling out each multiplication as the number of units by which a number measures another. Thus, "*gd* measures (*numerat*) *hz* by as many units as there are in *ab* whereas *ge* measures *kl* by as many units as there are in *ab* and *ed* measures *ml* by as many units as there are in *ab*". From here the conclusion is reached that the addition (*conjunctio*) *gd* measures *km* by as many units as there are in *ab*, and hence the number *km* equals the number *hz*. Translating back into "multiplication", al-Nayrīzī concludes that the area that is obtained from *ab* and *gd* equals the addition of the two areas that are obtained from *ab* and *ge* and *ab* and *ed*. And this is what we wanted to prove.

Thus we see that, while attempting to incorporate these results into the corpus of arithmetical knowledge displayed in the *Elements*, al-Nayrīzī nevertheless abode by the basic separation of realms. He did not import into the arithmetic books the kind of geometric reasoning with continuous magnitudes used by Euclid in Book II, but rather developed a proof that followed Euclid's own constraints for dealing with discrete quantities. Recall my explanation above in relation to V.1-V.6, concerning the conception of addition not as a binary operation, but rather as a gathering together of multitudes of instances of a given magnitude. Al-Nayrīzī deals here with discrete magnitudes and his proof is based on implicitly rearranging, according to the need, the instances of the magnitudes that appear in the said multitudes. As we will see, such rearrangement are performed explicitly in the work of Campanus as the basis of some of his arguments.

Al-Nayrīzī also formulated arithmetical equivalents of II.2-II.3. While in his version of Book II, as I stressed above, he had relied as did Heron on II.1 for proving these two propositions, here he went along with Euclid in the sense that he did not rely on the arithmetic version of II.1, but rather rehearsed the (now arithmetic) argument introduced for the respective version of II.1. Likewise, he also proved the arithmetic version of II.4 by a repeated application of II.2 (pp. 205-207). Thus, al-Nayrīzī clearly wanted to stress the autonomous, purely arithmetic characters of these propositions when presented in his comments to IX.16.

Also al-Nayrīzī's comments to propositions V.1-V.2 are interesting for our discussion here. First, concerning V.1, he indicated a possible difficulty in the argument of the proof (pp. 169-170). In order to see what he had in mind, consider the accompanying diagram which is the following:

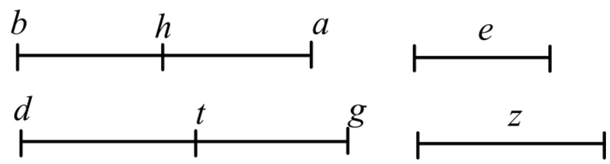


Figure 21: Al-Nayrīzī's diagram for Euclid's *Elements* V.1

Recall that in the argument of the proof, *ba*, *dg* represent equimutiples of *e*, *z* respectively, and it is required that the each of the latter be cut from each of the former, respectively. Now, in the simplest cases, this raises no difficulties. For example, if the two given magnitudes *ab* and *gd* are lines, he wrote, then, in order to do so, we can invoke Euclid's I.3 ("Given two unequal straight lines, to cut off from the greater a straight line equal to the less"). In the case where the magnitudes are arcs, al-Nayrīzī invoked Book III as providing the necessary justification. Most likely he had in mind a combination of propositions such as the following two:

> **III.27:** In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.
>
> **III.34:** From a given circle to cut off a segment admitting an angle equal to a given rectilinear angle.

Also for the case when the magnitudes are arcs al-Nayrīzī declared that no problem arises, but he did not explain why. It is also plausible that his implicit justification for this case might have been related, via arcs of circles, to the same two results of Book III, in conjunction with the following one:

> **VI.33**: In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences.

But for the case when the magnitudes involved are bodies, al-Nayrīzī indicated that the necessary operation of subtraction becomes "impossible" ("… *tunc illud erit impossibile*."). Nevertheless, he asserted, the existence of multiples is assumed in this case, only in order to imagine that if the number of times that *e* measures *ab* is two, then the number of times that *z* measures *gd* is also two, or if it is half of this then also *z* is half *gd*, and so on for any multiplicity whatsoever.

Thus, al-Nayrīzī considered this proposition as embodying several different, but *specifically geometric* situations, each of which required its own kind of justification. In other words, he was not thinking of "magnitudes" as a completely general concept on which we can argue in abstract terms, without specific justification for each case. Properties of equimultiplicity, so it seems, were for him differently rooted in basic properties specific to each kind of magnitude that can be considered.

His comments on V.2 are somewhat cryptic and we can at best conjecture what was that he had in mind. Al-Nayrīzī asserted that "there is nothing at all except the order of the branches of knowledge, of which the first is arithmetic, which is about numbers, and after which comes geometry. The proposition therefore demonstrates the basic, necessary principles which we will discover in this theory" (p. 170). In trying to understand what he had in mind here recall the original formulation of the proposition, which reads as follows:

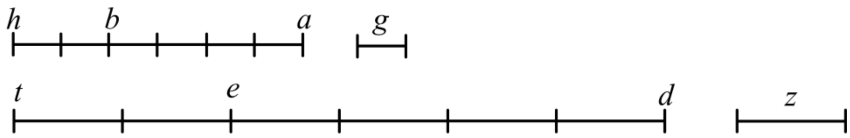


Figure 22: Al-Nayrīzī's diagram for Euclid's *Elements* V.2

The argument reads as follows:

> Once we know that *g* measures *ab* according to the number of times that *z* measures *ed* then the times that *g* measures *ab* and that it measures *bh* equal the number of times that *z* measures *de* and *z* measures *et*. Hence, the enumeration of multiples *ah* equals the enumeration of multiples *dt*, and this is what we wanted to prove.

As we just saw, in his treatment of V.1 al-Nayrīzī found it relevant to speak about the meaning of the proposition with respect to various kinds of geometrical magnitudes. Now in discussing V.2 he invoked the importance of "the order of the branches of knowledge" and then limited himself to a general argument presumably valid for all kinds of magnitudes. Perhaps he meant to say that it is not necessary to discuss the various cases separately because his argument covers "the basic, necessary principles that we will discover in this theory". Admittedly, this conclusion is somewhat conjectural and hence inconclusive. Much less can we know how later readers interpreted it, or if they paid attention to this remark at all.

4.2. Abu Kāmil

In Abu Kāmil's treatise on algebra, we find an arithmetical distributivity-like law of multiplication over addition for numbers that is grounded on essentially geometric considerations. This way to handle a basic property of an arithmetical operation reflects a basic tension permeating the entire treatise that is, at the same time, illustrative of broader issues arising in the Islamic mathematical traditions. Such issues derive from the attempt to reconcile diverging conceptions found at the sources of these traditions. While the numbers handled in the arithmetical parts of the *Elements* were always positive integers, the practical traditions from which Islamic arithmetic arose were at ease, from very early on, with fractions and irrational roots. Al-Khwārizmī's codification of the basic techniques for solving problems involving the square of an unknown quantity drew on Euclid-like geometrical arguments for obtaining their legitimation. Abu Kāmil attempted to provide a more systematic account of Al-Khwārizmī's techniques while making more evident the Euclidean source of the arguments. But the gap between Euclid's arithmetic and the more general views on numbers underlying Islamic mathematics could not be bridged simply by identifying numbers with Euclid's magnitudes. For one thing, while the operations of multiplication, division, or root extraction are closed in the domain of numbers, in the realm of magnitudes they involve a change in dimension. Thus, representing arithmetical operations geometrically involved serious conceptual as well as technical limitations and challenges. These challenges are evidently present in Abu Kāmil's handling of arithmetical rules, including those related with distributivity properties.[9]

[9] Alongside the historical context for Abu Kāmil's book as described in [LC1, 655-661], and the secondary sources cited there, I refer the reader to (Oaks 2011) for further details on these issues.

An example of particular interest for us here appears in Abu Kāmil's discussion of rules for multiplying expressions involving the unknown or its square. Results and methodological approaches taken from the arithmetic parts of the *Elements* are interestingly combined here with results and proofs originally meant to deal with continuous magnitudes, such as such as those of Book II. Thus, for instance, the case "How much is 10 and one thing times the thing", or its parallel, "How much is 10 minus one thing times the thing". Retrospectively seen, these embody the symbolic expressions $(10 + x)\cdot x = 10x + x^2$, or $(10 - x)\cdot x = 10x - x^2$. Abu Kāmil's proofs of these cases consist, essentially, in referring to a diagram where the situation is geometrically represented (Sesiano 1993, 342):

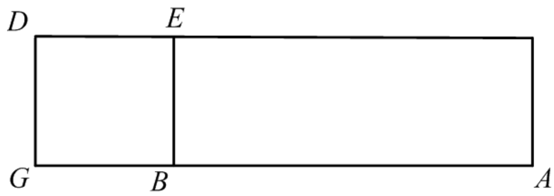


Figure 23: Abu Kāmil's diagram for solving a problem with squares of the unknown

Here *AB* is taken to be 10 and *BG* to be the thing, while the rectangle *AD* is the product whose value we are looking for. The figure is so constructed that segments *BE* and *GD* are equal, while also *GD* and *GB* are equal. Hence *BE* equals *GB*, which is the thing. Accordingly, then, *BD* is the square of the thing. Hence the rectangle *AD* is ten things and a square (of the thing), as stated in the rule.

Now, we saw above that in Euclid's proof of II.1 the distributivity of rectangle-formation for invisible figures derived from a geometric property (concatenation of rectangles) that is evident for visible figures. What we find here is a discussion of a rule of manipulation for the unknown quantity and its square (which are numbers). This arithmetic rule, however, is justified with the help of a geometrical argument. This justification, moreover, is embodied in a situation evidently similar to that of

Euclid’s II.1. The European readers of the Latin version of Abū Kāmil’s treatise became widely acquainted with this way of handling distributivity-like properties in the arithmetical context, that is, as a property that is conveniently backed by some kind of geometric justification. We shall find this approach repeated in many of the texts discussed below.

5. *Distributivity-like Results in the Latin Middle Ages*

The Latin medieval authors that I discuss in this section drew their views on Book II and on the other distributivity-like results found in Euclid’s *Elements* from a variety of sources. These include the various Latin versions of the *Elements* [LC1, 663-666], in the first place. They also included various Arabic and Hebrew texts that were translated and circulated in Latin Europe, such as Abraham bar Ḥiyya’s *Liber Embadorum* [LC1, 667-674]. Basic conceptions about numbers had already undergone, at this point, significant changes relative to those originally underlying Euclid’s work. For one thing, as already mentioned above, Islamic mathematics introduced and freely used fractions and even irrational squares in contexts where Euclidean propositions and methods were also invoked. The distributivity-like properties on which I focus here, appearing in the medieval texts discussed right below, were handled very often as part of conscious attempts to clarify the foundations of arithmetic and to provide, for this mathematical field of knowledge, the kind of axiomatic foundations that geometry had enjoyed for generations, at least since the time of the *Elements*. In many cases, discussions related with distributivity-like properties played a central role in such attempts, and for this reason they are certainly worthy of close historical attention.

5.1. *Liber Mahameleth*

Liber Mahameleth is a text on commercial arithmetic, presumably written in or near Toledo around 1143-1153 [LC1, 675-677]. It is of particular interest for our discussion here because its preliminary section is explicitly devoted to presenting what the author saw as the foundational rules of arithmetic. These rules are presented from an original perspective that combines arithmetic and geometric considerations, and that betrays the kinds of concerns faced by an author trying to come to terms with the actual source of their validity.

The preliminary section of *Liber Mahameleth* comprises eighteen propositions: the first eight (LM-1 to LM-8)[10] are arithmetic in contents and style, while the last ten (LM-9 to LM-18) are adaptations of results adopted from geometry. The former comprise several results taken directly from Abu Kāmil's *Algebra*, including associativity of the product. The latter comprise results from Euclid's Book II. Proposition LM-9, which is the equivalent of Euclid's II.1, provides the basis for proving all the following ones.

The author is aware that a reader with some knowledge of Euclid may object the order of the propositions and the reliance, for proving propositions that appear earlier in the

[10] In the critical edition of Vlasschaert (2012), the propositions are not numbered, but in the introductory text she provides algebraic renderings and numerates each corresponding formula. For ease of reference I am adding here a corresponding numeration for the propositions, with the initials LM, and I also refer to the corresponding page in the critical edition. A more recent, and in some senses more comprehensive, edition is that of Sesiano (2014). The latter had not been published when I completed [LC1]. Hence, in the interest of compatibility with my own previous article I will continue to refer here to the text as it appears in (Vlasschaert 2012). Notice that Sesiano also uses a somewhat different numbering of the propositions.

nature order of the *Elements*, on propositions that appear later on. Thus he wrote (Sesiano 2014, 597-598):[11]

> We deemed it appropriate to add next what Euclid stated in the second book, in order to explain with respect to numbers what he himself explained with respect to lines. It will be necessary for their proof to use certain propositions from the seventh book, for Euclid only spoke about numbers in the seventh book and the two following ones. For this reason Euclid should first be read and known thoroughly before embarking upon the present treatise on mahameleth.

Being this the case, the way in which the proposition is proved here is completely original and worth of attention. In the first place, the accompanying diagram is different from those appearing in any other medieval text for a version of II.1. It looks as follows:

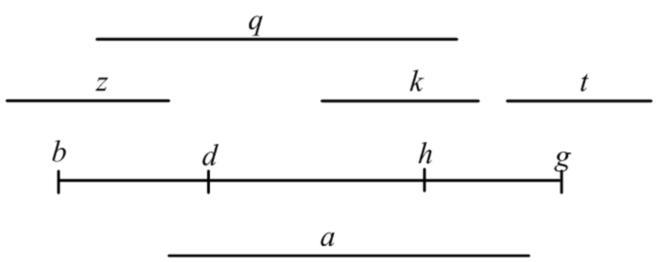


Figure 24: *Liber Mahameleth* LM-9

Like in Euclid's arithmetic books, the lines labeled with letters represent numbers but, unlike in Euclid's, these numbers can be fractions or even roots. This means that

[11] Sesiano comments in a footnote to this passage that, in spite of the warning, the only propositions on which are used in the following proofs are VII.17 and V.24. In p. 593, footnote 56, Sesiano explains that actually VII.17 and VII.18 are sometimes intercheangably referred to.

arguments in which the units are counted and possibly rearranged will not work smoothly as was the case with Euclid or even al-Nayrīzī. The proof combines ideas relating to proportions of both numbers and continuous magnitudes, and makes crucial use of V.24. As we saw above, this proposition expresses a distributivity-like law for proportions. But at the same time, the proof also relies on VII.18, which connects the operations of multiplication and ratio formation for numbers. If we allow ourselves here a symbolic rendering, for the sake of brevity, the property embodied in VII.18 would be the following:

> ***VII.18***: $b:c :: ab:ac$.

The details of the proof of LM-9 are as follows:

> In the figure above, two numbers a and bg are given, and bg is divided into parts bd, dh and hg. The proposition states that the product of a by bg equals the products of a by bd, a by dh and a by hg taken together. Further, q represents the product of a by bg, z the product of a by bd, k the product of a by dh, and t the product of a by hg. The proof can be rendered schematically with the symbolism of proportions (not used in the text, of course). It goes as follows:
>
> (d.1) $a \cdot bg = q$; $a \cdot bd = z$; $a \cdot dh = k$; $a \cdot hg = t$
>
> (d.2) By Euclid VII.18: $z:q :: bd:bg$; $k:q :: dh:bg$
>
> (d.3) By Euclid V.24:[12] $z + k : q :: bh:bg$
>
> (d.4) By Euclid VII.18: $t:q :: bd:bg$
>
> (d.5) By Euclid V.24: $z + k + t : q :: bd + dh + hg : bg$
>
> (d.6) But $bd + dh + hg = bg$, hence $z + k + t = q$ Q.E.D.

[12] In the Vlasschaert edition, p. 26, there is no direct reference to V.24, but just "*sicut euclides dixit in quinto*". (Sesiano 2014, 597-598) makes clear that the references are to VII.18 and V.24, as I indicate here.

Two important points should be stressed about this proof. Euclid's proof of VII.18 specifically depends on counting units (via VII.15). The author, however, applied it here to segments, and he did so without any further comment. The segments represent numbers, that's true, but as already stated the numbers here are not only integers and hence an argument based on counting units is problematic. Secondly, by relying on V.24, the author was actually grounding an arithmetic property on a property derived from the Eudoxian theory of proportions. Moreover, he did so via a result of Book II. On the face of it, once the author decided to rely on Book V, he might have followed the more straightforward approach of invoking V.1 (or one might also think of VII.5-6). Indeed, recall that V.1 embodies a law of distributivity of the product over the addition of several (not just two) magnitudes. But on closer look, multiplication in V.1 could be retrospectively seen now as concerning repeated addition of a magnitude to itself a number of times, whereas Euclid's II.1 (and hence also LM-9) is not meant as referring to that kind of multiplication. The "numbers" referred to in LM-9 are multiplied by another "number" rather than repeatedly added to themselves, as in V.1. Also in this regard, then, the mixture of domains and approaches is quite unusual. It may well be the case that the author realized that the intended meaning of "multiple", when multiplying by segment *a*, required repeated reliance on V.24 rather than a direct application of V.1. But then again, this is somewhat roundabout since also VII.18 is applied and since the basic intention was to prove a property of numbers.

Upon examining the entire preliminary section more closely, one comes across an even more complicated picture, given that two special cases of distributivity-like properties are already proved in the book previous to LM-9. This is the case with the purely arithmetic propositions LM-6 and LM-7, which deal with the relationship between *division* and, respectively, subtraction and addition. Also the details of these proofs are quite interesting for reaching a clearer insight into the rationale of this preliminary section. The first four propositions in the section comprise statements and proofs of elementary arithmetic properties such as associativity of multiplication or division of three or four numbers (with strong references to Abū Kāmil). The next proposition, LM-5 (pp. 20-22), is needed for proving LM-6. Its enunciation is the following:

> **LM-5:** If six numbers are given, such that the first is to the second as the third is to the fourth, and the fifth is to the second as the sixth is to the fourth, then that by which the first supersedes the fifth or is exceeded by the fifth is to the second as that by which the third supersedes the sixth or is exceeded by the sixth.

This is a modified version of V.24, when *subtraction* is involved instead of addition. As already mentioned, Robert Simson remarked that Euclid's proof of V.24 can be easily modified to obtain the result stated here in LM-5, namely by relying on V.17 rather than on V.18 (Simson 1804, 119). Indeed, this is precisely what the author of *Liber Mahameleth* did in his proof. Let us consider the details.

The six given numbers are *ab*, *g*, *dh*, *z*, *ak*, *dt*, and they define three proportions *ab*:*g* :: *dh*:*z*, and *ak*:*g* :: *dt*:*z*.

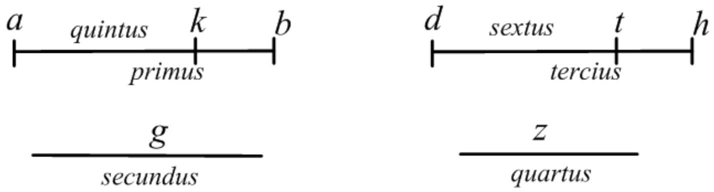


Figure 25: *Liber Mahameleth* LM-5

The proposition then states that *kb*:*g* :: *th*:*z*, and the proof can schematically be summarized in the following steps:

(e.1) from *ak*:*g* :: *dt*:*z* we obtain *g*:*ak* :: *z*:*dt*

(e.2) Now, from *ab*:*g* :: *dh*:*z and g*:*ak* :: *z*:*dt* it follows that *ab*:*ak* :: *dh*:*dt*

(e.3) From *ab*:*ak* :: *dh*:*dt* it follows that *bk*:*ak* :: *ht*:*dt*

(e.4) Finally since *ak*:*g* :: *dt*:*z* and *bk*:*ak* :: *ht*:*dt* it follows that *bk*:*g* :: *ht*:*z*

QED

Some minor remarks on the proof are in order:

- The author justified step (e.1) by invoking Euclid's V.16, but from the latter proposition and from *ak*:*g* :: *dt*:*z* what we obtain is *ak*:*dt* :: *g*:*z*.
- Steps (e.2) and (e.4) rely on V.22, which is not mentioned by the author.
- The crucial step is (e.3), which, as Simson said, depends on V.17. The author wrote here "*Secundum proportionalitatem*".
- Notice that both V.17 and V.22 are proved in Euclid directly from the Eudoxean definition of proportion. There are no parallels to these two propositions in Book VII. Thus, also in this sense the author is relying here directly on Book V for his foundational result.
- Whereas in the enunciation, the possibility "or is exceeded" (by the fifth/sixth) is mentioned, there is no reference to that in the proof.

The proof of LM-6 (pp.22-23), which handles *subtraction* rather than addition, is a direct application of LM-5.

Referring once again to the above diagram, two numbers are given, *ab* and *ak*, *kb* being their difference. When divided by *g*, they yield, respectively, *dh* and *dt*, and *th* is their difference. The proposition thus states that if the difference *kb* is divided by *g*, the result is *th*, the difference of the divisions.[13] The argument of the

[13] On p.23 of the Vlasschaert critical edition there is an additional diagram,

proof is simple: from $ab/g = dh$ it follows that $dh \cdot g = a$ and hence $g:ab :: 1:dh$. Likewise, one can deduce $g:ak :: 1:dt$. Here one can apply LM-5, from whence $g:kb :: 1:th$, and from here, the desired results follows easily.

The author indicated that the following proposition is similar, but now for the case of addition (rather than subtraction as in LM-6). Therefore one needs a rule that is similar to LM-5, but applies to addition. But such a rule, he stated, had been already proved by Euclid and hence it would not be necessary to prove it in the treatise again. The reference is, of course, to V.24. Thus, LM-7 is formulated as follows (p. 24):

> **LM-7:** When any two numbers are divided by another number, then the outcomes of both divisions taken together equal the result of dividing by the same divisor both numbers taken together.

The diagram is the following:

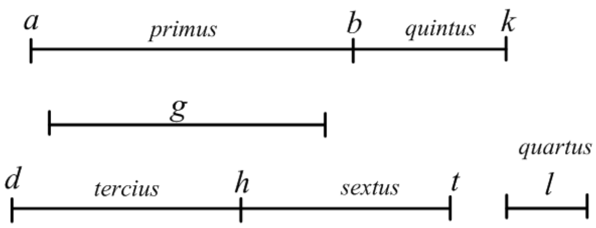
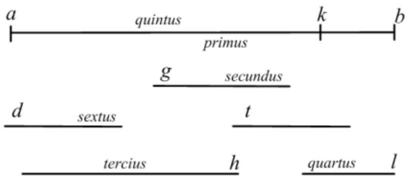


Figure 26: *Liber Mahameleth* LM-7

And the proposition states that if we divide *ab* by *g* and the result is *dh*, and if we divide *bk* by *g* and the result is *ht*, then, the result of diving *ak* by *g* is *dt*. The proof is similar to that of LM-6, but in its crucial step it relies directly on Euclid's V.24.



It does not seem, however, to fit the argument presented in the proof.

Liber Mahameleth comprises many results that are illustrative of the challenges faced by medieval authors trying to come to terms with the very idea of providing foundations for arithmetic. Distributivity-like results are among such results. Of particular interest is the way in which an underlying tension manifests itself, arising from the attempt to understand the basic properties of numbers without thereby giving away the traditional, Euclidean centrality of geometry as the field that is better understood and axiomatically founded.

5.2. Fibonacci

Yet another illuminating perspective on the interest in distributivity-like properties is found in Fibonacci's *De Practica Geometriae*, composed in 1220 or 1221.[14] As in the more famous *Liber Abaci*, also in this treatise, Fibonacci used propositions from Euclid's Book II for solving problems involving squares of unknowns. But unlike in *Liber Abaci*, there is an entire section in the *Practica* where Fibonacci directly cited twelve propositions from Euclid's *Elements*, the first nine of them being versions of propositions in Book II (II.8 is absent). He introduced original arguments for some of the propositions. For details see [LC1, 677-684]. My point here is that in all of these propositions, the kind of distributivity afforded by II.1 (or its direct derivatives, such as II.2-II.3) plays a central role. Fibonacci presented these results in two different versions. First, an arithmetic version that applies to "any number" and is proved by counting units. Secondly, a more geometric version in which "a straight line" is divided into segments as in II.1. For the second kind of statements, he typically provided no proof. The proofs given by Fibonacci deserve some further analysis.

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[14] See (Moyon 2012), for a broader discussion of the genre of *Practica Geometriae* and its relation with the early development of algebra.

The arithmetic version of II.1 appears in Fibonacci's proposition PG-31. Here the number *ab* is divided into *ag*, *gd*, and *db*, and Fibonacci claims that the products of *ag* by *ab* together with *gd* by *ab* and *db* by *ab* equal the product of *ab* by *ab* (Hughes 2008, 26-27).

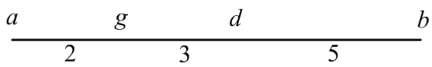

Figure 27: Fibonacci's diagram for the arithmetic version of Euclid's *Elements* II.1

The reason adduced is, simply, that "the number of units in part *ag* with those in *ad* will produce the product of *ag* by [*ad*]."[15] And the same is said for the other two pairs. Hence, Fibonacci concludes, "because there are as many units in the number *ab*, namely in the parts *ag*, *gd*, *db*, so many are united in the number *ab* from the multiplication of *ag*, *gd* and *db*, by *ab*." And on the other hand "as many units as there are in ab, so many arise from the multiplication of ab by itself". His argument, then, is similar to that found in Euclid's VII.5. After concluding that both products are equal, as required, Fibonacci also gave a numerical example.

In PG-32, Fibonacci divides a "straight line" into many parts and multiplies each part by "another line", and then states that "the sum of all the products equals the product of the whole divided line by the other line". Here no proof is provided, but just a numerical example of 2,3, and 5 multiplied by 12. PG-33 is version of II.3 and PG-34 is of II.4. In both cases the enunciation is for "straight lines", rather than for numbers, and the proof is based on the previously proved results on

[15] I am relying here on Hughes' translation and I have no direct access to the original. In this specific part of the proof, *ab* and *ad* are interchanged in two places, and I am assuming that this is a typo. Likewise, I assume that Fibonacci's conclusion that follows immediately, and which sounds to me as a somewhat weird type of mathematical reasoning, is indeed what the original says (or that at least it is close enough to it).

distributivity. In both cases Fibonacci added that the result "can be shown with numbers".

Fibonacci's statements of the propositions allowed considering them simultaneously as related to both geometry and arithmetic, and he could move quite freely from one realm to the next when necessary. Some of the geometric results he relied on he took for granted, some he just illustrated with numerical examples. He also introduced new proofs for the Euclidean results in which he tried an innovative, proto-algebraic reasoning as part of the argument. This flexible view of the relationship between geometry and arithmetic certainly was conveyed, at least at the implicit level, together with the more concrete results and techniques explicitly taught in his treatises. Given the rather wide audiences that these treatises reached over the next centuries, his approach no doubt played a significant role in eventually weakening the strict separations of realms typical of Euclid's practice in the *Elements*.

5.3. Jordanus Nemorarius, Campanus de Novara

The two most prominent medieval authors who were involved with questions pertaining to the foundations of arithmetic were Jordanus Nemorarius and Campanus de Novara. Both were active in the 13th century. Because of the originality of their ideas and the influence they exerted on mathematicians of later periods of time, they deserve a detailed, separate discussion concerning the ways in which distributivity-like properties appear in their works.[16] For the sake of completeness in the

[16] See (Corry 2016) for such a detailed discussion.

presentation here, I will summarize now some of the most important points related to this issue.

In his treatise *Arithmetica*, Jordanus sought to achieve for arithmetic what Euclid had done for geometry, in all what concerns the derivation of the body of arithmetic from definitions, postulates, and common notions. Moreover, he explicitly avoided reliance on geometrical concepts or results of any kind when putting forward his project [LC1, 684-689]. A central role was accorded in this treatise to five distributivity-like results, respectively parallel to Euclid's VII.5, VII.6, V.1, and (in two different versions) II.1. The first three of these are, like their Euclidean counterparts, statements of the type "if … then" (see above). The last two embody a property similar to left- and right-distributivity of the product over addition for numbers. Interestingly, in Jordanus' treatise we find no proposition that is parallel to Euclid's V.2, following the one that is parallel to V.1.

In order to convey the overall feeling of Jordanus' style in handling these propositions, it seems convenient to focus on the pair of propositions that is parallel to Euclid's II.1. These are the following: [17]

> **A-I.9:** That which is obtained by multiplying any number by as many as one pleases equals that which is obtained by multiplying the same [number] by their combination [i.e., their sum].
>
> **A-I.10:** That which is obtained by multiplying as many numbers as one pleases by some number equals that which is obtained by multiplying their combination [i.e., their sum] by the number.

[17] For easiness of reference I use here a numeration of the propositions which does not appear in the original.

If we allow ourselves a symbolic rendering of the two for the purposes of illustration we obtain the following:

***A-I.9*:** $a \cdot b + a \cdot c + a \cdot d + \ldots = a \cdot (b+c+d+\ldots)$,

***A-I.10*:** $b \cdot a + c \cdot a + d \cdot a + \ldots = (b+c+d+ \ldots) \cdot a$.

Notice that whereas A-I.10 embodies a legitimate arithmetic operation (a sum of numbers being multiplied by another number) a parallel proposition stated in the purely geometrical context of Euclid's Book II would make little sense. Indeed, such a parallel formulation would amount to something similar to the following:

> **II.1′:** If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by each of the segments and the uncut line.

It would not be clear, in the first place, what is the sum of rectangles that such a formulation would involve, given that in the context of Book II the sums of rectangles are concatenations, one to the right of the previous one. Moreover, to the extent that one can make sense of this formulation, it does not seem to add anything of real content to the original II.1.

The arithmetic case is more interesting, and apparently more meaningful, but at the same time, Jordanus decision to include A-I.10 is quite remarkable given that in A-I.8 he had proved the commutativity of the product in general:

> **A-I.8:** If two numbers are multiplied alternately, the same number is obtained in both cases.

Why did Jordanus nevertheless prove A-I.10, then, if it follows from applying A-I.8 to A-I.9? In order to answer this question one would have to look at the details of the proofs of all these propositions (as explained, e.g., in (Corry 2016)). The details indicate that propositions A-I.9-A-I.10 actually represent not just two different versions of the same situation, but actually two propositions that were truly different for Jordanus: a “number” is multiplied by “as many numbers as one pleases”, and “as many numbers as one pleases” are multiplied by “a number”.

In addition to these five propositions, some of the subsequent ones in the preliminary section of the *Arithmetica* embody additional cases of distributivity-like properties of various kinds. These are, as a matter of fact, versions of II.1-II.3, that generalize or provide particular cases of the previous ones, and that are proved on while relying on those previous ones, particularly on A-I.9. Thus, for instance, the following three (of which I just give the symbolic rendering):

***A-I.11*:** $(a + b + c + \ldots)\cdot(p + q + r + \ldots) =$
$a\cdot p+ a\cdot q+ a\cdot r+ \ldots + b\cdot p+ b\cdot q+ a\cdot r+ \ldots + c\cdot p+ c\cdot q+ c\cdot r+ \ldots$

***A-I.13*:** If $a = b + c + d + \ldots$ then $a\cdot a = a\cdot b + a\cdot c + a\cdot d + \ldots$

***A-I.14*:** If $a = b + c$ then $a\cdot b = b\cdot b + b\cdot c$

Jordanus’ conscious attempt to provide a rigorous presentation of arithmetic, not found in previous treatises, led him to make a clear distinction between repeated addition of numbers, and multiplication of two numbers, but at the same time to present these two ideas as closely related. A main focus of attention in pursuing that distinction appears in relation with distributivity-like results, and a possible reason for this is that in those treatises where he learnt his arithmetic, he did not find a satisfactory treatment of such results.

Campanus de Novara published a Latin version of the *Elements* around 1260. His treatment of Book II does not differ from the standard, Euclidean one found in other medieval versions of the treatise [LC1, 689-692]. His treatment of distributivity-like results in the framework of Books V and VII, however, is highly original. In the preliminary section to Book V, Campanus introduced lengthy additions and comments. He did not hesitate to explain to his readers what, in his view, was in Euclid's mind when writing this or that definition. Campanus also commented on the highly difficult character of the theory of proportions as presented in Book V, while stressing explicitly that these difficulties arise mainly from the need to deal, within one and the same framework, with irrational as well as with rational ratios (Busard 2005, 173-175). As I already explained in [LC1], Campanus devoted some efforts to discuss, echoing Jordanus, the significance of Euclid's double treatment of proportions, once for continuous magnitudes and once for numbers.[18]

In order to help the reader in following and understanding the arguments of the proofs in Book V, Campanus associated numbers to the segments that appear in the diagrams (he did the same in many diagrams of Books VII-IX as well). Jordanus had followed a similar approach, and it was quite natural that he did so as part of his treatment of results in the arithmetical books. It seems much less natural, however, to find this done in the case of Campanus, given his stress on the essential difference between handling proportions that involve continuous magnitudes and those that are purely arithmetic. Let us see how this appears, for instance, in the diagram accompanying V.1, which is the following (p. 177):

[18] Campanus' discussion of proportions also raises some additional issues concerning both textual and conceptual difficulties, but they are beyond the scope of this article. See (Rommevaux 2007).

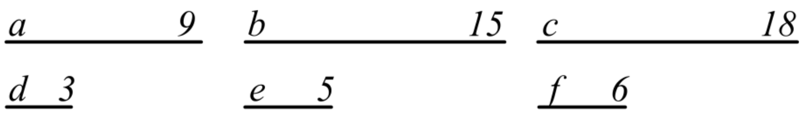

Figure 28: Campanus' diagram for Euclid's *Elements* V.1

Here the three magnitudes a,b,c are said to be equimultiples of d,e,f respectively, and the proposition states that $a + b + c$ is the same equimultiple of $d + e + f$ as a is of d. The numbers appearing in the diagram are not even mentioned in the specification or in the proof, but one can imagine that they may have helped the reader follow the argument.

Also in the introductory section to Book VII Campanus devoted focused attention to distributivity-like properties. Thus, one of his common notions, which is fundamental to the attempted systematic foundation of arithmetic, states that if the unit is multiplied by any number or if a number is multiplied by the unit, then the result is the number itself (Busard 2005, 231). Campanus added three additional ones, which I ask to include here under the category of "distributivity-like". They are the following:

- Any number that measures two numbers, measures also their sum.
- Any number that measures some number, measures also any number measured by it.
- Any number that measures the whole and the deducted, measures also the remainder.

Intrinsically related with the attempt to provide an axiomatic foundation for arithmetic is the idea of the autonomy of such a body of knowledge vis-à-vis the other parts of mathematics presented in the *Elements*. Campanus consistently stressed this issue throughout the arithmetic books, and in particular he stressed the autonomy of proofs in Book VII vis-à-vis those of Book V. As I already explained in [LC1], Campanus explained that the "*propria principia*" of the two books are different and, hence, corresponding propositions should be proved separately, and based on those specific principles alone in each case. In relation with the proof of VII.5, for instance, we find the following statement (p. 235):

> Euclid wanted that the arithmetical books would not have to rely on the previous ones, but rather that they would stand by themselves, and results that he proved in the fifth books for quantities in general he proved here for numbers in this fifth of the seventh.

When examining the proofs in some detail, however, we notice that in some cases this autonomy did not go beyond repeating, while fully rewording for numbers, an argument already presented in Book V.[19]

The most important point to mention here in relation with Campanus' version of the *Elements* concerns a collection of fifteen commentaries added after the proof of IX.16. As already explained in [LC1][20], these commentaries comprise, among other things, arithmetic versions of propositions from Book II that embody interesting distributivity-like properties. I also mentioned above an interesting peculiarity of Euclid's proof of IX.15, namely, that it transgressed the self-imposed rules of separation, and used arithmetic versions of II.3-II.4, without further comment. Al-Nayrīzī in his own commentary, in turn, formulated IX.16 as a generalized version of IX.15, and added in relation to it his own arithmetic versions of II.1-II.4. Now Campanus, following on al-Nayrīzī's footsteps, also added here his own commentaries.

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[19] In other Latin versions of the *Elements,* instead of such a repetition, often there is just a direct reference to a corresponding proposition in Book V. See Busard 2005, 560.

[20] See also footnote **Error! Bookmark not defined.** above.

A reader of Campanus who was also acquainted with Jordanus' *Arithmetica* (if there was any) would have easily recognized the close relationship (sometimes verbatim repetition) of Campanus' comments and Jordanus' basic rules of arithmetic discussed above. Book VII opened by rehearsing Jordanus' attempt to providing an axiomatic foundation, yet Campanus was also using the opportunity to include those elementary propositions that Jordanus had developed in the first chapter of his book. But as with Book VII, some of the technical changes that Campanus introduced in his presentation lead to some noteworthy differences. Thus, for example, Campanus' first two comments state two symmetric, distributivity-like rules involving products and additions. Their enunciations are parallel to, but not identical with, Jordanus' A-I.9 and A-I.10. This is how they appear in the text:

> **C-IX.9-1:** That which is made by multiplying a number by as many as we wish equals that which is made by multiplying it by them.
>
> **C-IX.9-2:** That which is made of as many numbers as you wish in one, equals that which is made by their sum on it.

Campanus' proof of the first one is based on directly applying VII.5. For the proof of the second one he applied to the first the commutativity of the product, a property that appears in Campanus VII.17 (or Euclid's VII.16).

Campanus stressed in relation with C-IX.9-1, that "the first of the second" (i.e.,Euclid's II.1) states the same thing but for lines. A similar statement appears in all the following commentaries, 4 to 12, with relation to each of the propositions II.2-II.10 respectively. The first three of these correspond to the distributivity-like properties II.2-II.4, and they are proved by direct application of the first two rules.

The important point to notice concerning these commentaries is that for the sake of their proof, Jordanus had had to start with a distributivity-like property for *multiplicities* (A-I.6). This provided the basis for proving other statements for arithmetic that are truly parallel to those of geometry, in the sense that they refer to a multiplication of *number by numbers*. Campanus, in turn, could base his proof directly on VII.5, which already handled the distributivity-like property of the multiplicities.

Campanus's version of the *Elements* had a decisive influence on the way that the treatise was read and understood over the following generations, particularly in relation with the issue of the relationship between arithmetic and geometry. This is of course also the case concerning distributivity-like properties. Of particular importance in this regard is the addition of purely arithmetic versions to Book IX. Readers of the Campanus version, or of any other work derived from it, would now have good grounds – and by all means better grounds than those of a reader of any previous treatise – for seeing these properties as inherently arising within the purely arithmetic realm, without any need for additional support coming from geometric considerations.

6. *An Example from Hebrew Mathematics: Gersonides*

In the introduction to *Sefer Maaseh Hoshev*, dating from 1321, Gersonides told to his readers that he would assume a thorough knowledge of the arithmetical books of the *Elements*, and that he would not prove in his own text any of the results appearing in those books. Among the most basic results that he did prove there are some fully arithmetic versions of propositions from Book II, and specifically those embodying distributivity-like properties. The proofs do not contain any new idea, but I think that it is interesting to take a closer look at the way in which they are presented here as purely arithmetic results with no trace of geometric origins or content. This can certainly be taken to represent the manner in which these kinds of results were already understood by the early fourteenth century. This is most clearly seen in Gersonides' version of Euclid's II.1 (MH-I.2), which reads as follows:[21]

> **MH-I.2:** When two numbers are given and one of them is divided into as many parts as we wish, the area of one of the numbers by the other equals the sum of the areas of each part of the one multiplied by the other.

The diagram is as follows (for convenience I use Latin letters instead of the Hebrew ones appearing in the original):

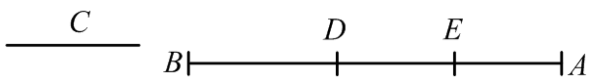

Figure 29: Gersonides' *Maaseh Hoshev* I.1

And the text of the proof is the following:

> For let the numbers *AB* and *C* be given, and let the number *AB* be divided into parts, *AE*, *ED*, *DB*. I say that the area of *AB* on *C* equals the area of *AE* on *C* and the area of *ED* on *C* and the area of *DB* on *C* taken together. Proof: the area of *AE* on *C* contains the number of units in *AE* as many times as *C*, the area of *ED* on *C* contains the number of units in *ED* as many times as *C*, and the area of *DB* on *C* contains the number of units in *DB* as many times as *C*. Therefore, all of them

[21] Gersonides consistently uses the term "area" (שטח) to indicate products of numbers. But other than this, there is no hint of any kind of geometric thought involved here. All quotations are taken from (Lange 1909).

> taken together contain the number of units of *AE*, *ED*, *DB*, taken together as many times as *C*. But the number of units of *AE*, *ED*, *DB*, taken together equals the number of units in *AB*.

The diagram is interesting because, on the face of it, it is similar to that of Heron (as well as others we saw above), but unlike with Heron, it actually accompanies here an arithmetic kind of proof based on mere counting of units. Gersonides speaks here of, for instance, "the area of *AB* on *C*", and as in Heron's proof this is not represented in the diagram. Unlike with Heron, the result (even though it is called "area") is yet another number, but it is *not* represented by another line in the diagram as was usually the convention for arithmetic-type proofs, where all numbers as well as all products of numbers mentioned in the proof appear as lines in the diagram.

The next two propositions are just extension of MH-I.2. Thus, MH-I.3 is similar to Jordanus' A-I.11, whereas MH-I.4 is parallel to Euclid's II.2:

> **MH-I.3:** When two numbers are given and each of them is divided into as many parts as we wish, the area of one of the numbers by the other equals the sum of the areas of each part of the one multiplied by each and every part of the other number.
>
> **MH-I.4:** If any given number is divided into two numbers, the area of the number by any of its parts equals the area of one part by the other, together with the square on the part.

Both are proved by direct application of MH-I.2, and they are used to prove some of the new propositions that Gersonides proves in the book.

We thus see that, as with any other issue addressed in *Sefer Maaseh Hoshev*, Gersonides' treatment of distributivity-like properties was consistently arithmetic throughout. This was perhaps representative of the way in which most of his contemporaries had come to conceive of the entire issue by this time.

7. *Concluding Remarks*

In concluding this article I can refer back to my final comments in [LC1]. The next historical stage in considering the history of presentations of distributivity-like results within the Euclidean tradition was already within the world of the printed text, which was inaugurated with the 1482 Ratdolt edition and later on consolidated with the Commandino edition of 1572. These printed versions differed in important senses from the medieval ones, among other things because of the strong influence of the Campanus version. The changing relationships between geometry and arithmetic and the vigorous trends of symbolic algebraic techniques that began to attract increased attention opened the way to additional perspectives on the treatment of more general ideas of distributivity and on their place in the overall economy of mathematical knowledge. These new perspectives are worth of further, detailed attention but of course they cannot be pursued here and I leave them for a future opportunity.

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